# Enumerating log rational curves on $\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^s \oplus \mathcal{O}(-a))$

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## Question

Let $X$ be a smooth, projective variety. Let $(C, p_1, \dots, p_n)$ be a general pointed curve of genus $g$. Let $\beta \in H_2(X, \mathbb{Z})$ be an effective curve class on $X$, and let $q_1, \dots, q_n \in X$ be general points.

Question: How many morphisms $f : C \to X$ are there in class $\beta$ (i.e. $f_*([C]) = \beta$) satisfying $f(p_i) = q_i$ for all $i \in [n] = \{1, \dots, n\}$?

## Geometric Tevelev Degrees

Let $X$ be a smooth projective variety over $\mathbb{C}$ of dimension $r$, and let $\beta \in H_2(X, \mathbb{Z})$ be a non-zero effective curve class. Fix $g, n \geq 0$ such that $2g - 2 + n > 0$ so that the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a Deligne-Mumford stack. Let $\mathcal{M}_{g,n}(X, \beta)$ be the open subscheme of smooth maps, and consider the forgetful map

$$\tau : \mathcal{M}_{g,n}(X, \beta) \to \mathcal{M}_{g,n} \times X^n.$$

$$(f : (C, \{p_i\}_{i\in[n]}) \to X) \mapsto (C, \{p_i\}) \times \{f(p_i)\}_{i\in[n]}$$

Assume that the expected dimension of $\mathcal{M}_{g,n}(X, \beta)$ is equal to the dimension of $\mathcal{M}_{g,n} \times X^n$ or equivalently,

$$\beta \cdot K_X^\vee = \dim(X)(n + g - 1),$$

and all the dominating components of $\mathcal{M}_{g,n}(X, \beta)$ are generically smooth of the expected dimension. Then the **geometric Tevelev degree** $\mathrm{Tev}^X_{g,\beta,n}$ of $X$ is defined as the degree of the forgetful map $\tau$.

Example: [1] For the case when

$$X = \mathbb{P}^1, \quad \beta = (g+1)[\mathbb{P}^1], \quad \text{and} \quad n = g + 3$$

we have

$$\mathrm{Tev}^X_{g,(g+1)[\mathbb{P}^1],g+3} = 2^g.$$

This result also follows from early work on Vafa-Intriligator formulas [2].

## Logarithmic Tevelev Degrees with tangency conditions

Let $X/\mathbb{C}$ be a smooth, projective toric variety, and let $D_\rho \subset X$ be the torus-invariant divisors, indexed by the maximal rays $\rho \in \sum(1)$ in the fan $\sum$ of $X$. The **boundary** of $X$ is given by $\bigcup_{\rho\in\sum(1)} D_\rho$ and its complement is the **interior** $X^\circ$ of $X$.

Let $\beta \in H_2(X, \mathbb{Z})$ be an effective curve class, and let $n \geq 3$. Assume that

$$\int_X \beta \cdot D_\rho \geq 0, \quad \text{for all } \rho \in \sum(1).$$

For each $\rho$, let $\mu_\rho = (\mu_{\rho,\nu})_{\nu=1}^{m_\rho} \in \mathbb{N}^{m_\rho}$ be a vector of positive integers, with sum $\int_X \beta \cdot D_\rho$. In the case when $\int_X \beta \cdot D_\rho = 0$, we let $\mu_\rho$ be the empty vector and $m_\rho = 0$. We write $m = \sum_{\rho\in\sum(1)} m_\rho$.

Consider $n + m$ distinct points

$$(\mathrm{P}, \mathrm{Q}) = \left( \{p_i\}_{i=1}^n, \{q_{\rho,\nu\rho\in\sum(1)}\}_{\nu=1}^{m_\rho} \right) \in (\mathbb{P}^1)^{n+m}.$$

Let $\mathcal{M}_\Gamma(X)$ denote the moduli space of log maps $f : (\mathbb{P}^1, \mathrm{P}, \mathrm{Q}) \to X$ such that

- $f_*[C] = \beta$,
- $f$ maps $p_i$ to $X^\circ$ and $f$ maps $q_{\mu,\nu}$ to the toric boundary of $X$ with multiplicity $\mu_{\rho,\nu}$, i.e.

$$f^* D_\rho = \sum_{\nu=1}^{m_\rho} \mu_{\rho,\nu} q_{\rho,\nu}.$$

We use $\Gamma$ to denote the data of $\beta$, the integer $n$, and the vectors $\mu_\rho$ specifying the tangency profile of $f$ along the boundary.

Assume that $n = \frac{m}{\dim X} + 1$, then we define the (genus 0) **logarithmic Tevelev degree** $\mathrm{logTev}^X_\Gamma$ to be the degree of the forgetful morphism

$$\tau : \mathcal{M}_\Gamma(X) \to \mathcal{M}_{0,n} \times X^n.$$

## Main Result [3] [C. Lian, S.]

Fix integers $r, s \geq 1$ and $a \geq 0$. Consider the space $X_{r,s,a} = \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^s \oplus \mathcal{O}(-a))$ and let $\Gamma$ denote the tangency data for maps to $X_{r,s,a}$. Assume that $n = \frac{m}{r+s} + 1 \geq 3$ is an integer, where $m = \sum_{j=0}^{r+s+1} m_j$ is the total number of distinct intersection points of a rational curve $\mathbb{P}^1$ with the toric boundary, as prescribed by $\Gamma$. Here $D_1, \dots, D_{r+1}$ are the pullbacks of the torus-invariant divisors on $\mathbb{P}^r$, $D_0$ is the divisor corresponding to the factor $\mathcal{O}(-a)$, and $D_{r+2}, \dots, D_{r+s+1}$ are the remaining torus-invariant divisors. Fix general points $p_1, \dots, p_n \in \mathbb{P}^1$ and $x_1, \dots, x_n \in X^\circ$. If

$$m_j \leq n - 1 \ \forall j = 1, \dots, r + s + 1, \qquad \sum_{j=r+2}^{r+s+1} m_j \geq (s-1)(n-1), \qquad m_0 + \sum_{j=r+2}^{r+s+1} m_j \leq s(n-1)$$

hold, then

$$\mathrm{logTev}^{X_{r,s,a}}_\Gamma = \left( \prod_{j=0}^{r+s+1} m_j! \right) \left( \prod_{j=0}^{r+s+1} \prod_{\nu=1}^{m_j} \mu_{j,\nu} \right) a^{\sum_{j=0}^{r+1} m_j - r(n-1) - m_0} \binom{\sum_{j=0}^{r+1} m_j - r(n-1)}{m_0}.$$

Otherwise, $\mathrm{logTev}^{X_{r,s,a}}_\Gamma = 0$. When $a = 0$, the expression $0^0$ is interpreted to equal 1.

## Enumerativity of $X = \mathrm{Bl}_{q_1,\dots,q_r}(\mathbb{P}^r)$

We showed that, for the case $X = \mathrm{Bl}_{q_1,\dots,q_r}(\mathbb{P}^r)$, the prediction of [4] for the logarithmic Tevelev degree does not always hold.

Example: Let $X = \mathrm{Bl}_{[0:1:0],[0:0:1]}(\mathbb{P}^2)$. Any map $f : \mathbb{P}^1 \to X$ is given by five sections $g_1, \dots, g_5$ organized as $f = [g_1 g_2 g_3 : g_1 g_4 : g_2 g_5]$.

- We proved that $\mathrm{logTev}^X_\Gamma = 2400$ when $\mu_1 = \mu_2 = (1), \ \mu_3 = (1, 1, 1, 1), \ \mu_4 = \mu_5 = (5),$ contradicting the prediction by [4] i.e. 5400.
- We have $\mathrm{logTev}^X_\Gamma = 1152$, as predicted by [4] $\mu_1 = \mu_2 = (1), \ \mu_3 = (1, 1, 1), \ \mu_4 = (4), \ \mu_5 = (2, 2).$

## Future Directions

The natural next direction would be to compute $\mathrm{logTev}^X_\Gamma$ for curves with genus $g > 0$.

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